Rigid Body Motions in Curved Finite Elements

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A method is described to correct the stiffness matrix of a curved element that is deficient in rigid body motion. The method is general and can be used for any curved element; an illustration is given using a cylindrical element.

Introduction

DISPLACEMENT functions used to construct the stiffness matrix of a finite element should possess the following properties.

1) Infinitesimal rigid body motions should be accurately represented. If this requirement is not met the conditions of equilibrium of the element are not satisfied.[1,2]

2) The displacement functions should contain all the lower terms of a complete set of functions.[3] This requirement insures monotonic convergence by mesh size reduction.

3) A minimum degree of interelement continuity must be maintained between adjacent elements. This minimum degree of compatibility must insure a perfect match for the inplane and the out of plane components of displacement. Also for the out of plane component, slopes tangent and normal to all common edges of two adjacent elements must match. This requirement then insures convergence to an exact result by mesh size reduction.

The importance of the last two requirements is firmly established; however, the first requirement has been shown to be problem dependent. If the structure to be analyzed is so constrained that no element of the structure is ever going to undergo any rigid body motion, then obviously this requirement can be violated. For example, axisymmetric elements acted upon by axisymmetric loads need to have only one rigid body mode: a rigid translation parallel to the axis of symmetry. For this particular type of element a truncated cone as used by Grafton and Strome[4] always includes a rigid body motion parallel to the longitudinal axis. However, if the axisymmetric element is to have curvature in the longitudinal direction, then all the rigid body modes are absent. Jones and Strome[5] recognized such a deficiency and reintroduced a longitudinal translation in their element. Later, Stricklin et al. reported on a similar improved element but omitted the longitudinal rigid body motion altogether.[6] This last element is capable of handling asymmetric loading, therefore it is not difficult to imagine a loading in which many elements would have to undergo considerable transverse motion; a cantilevered structure would lead to such a situation. Haisler and Stricklin[7] studied the influence of longitudinal translation and observed that such a rigid motion is recuperated by mesh size reduction.

For elements of rectangular aspects, Bogner, Fox, and Schmit[8] developed a systematic method for constructing acceptable displacement fields. However, for curved cylindrical elements, only two rigid body modes are accounted for. The same authors reported on a (48 × 48) stiffness matrix and mentioned that an eigenvalue analysis of such a matrix indicated that rigid body motions were adequately represented.[9] However, as pointed out in our study of curved cylindrical elements, rigid body motions cannot be represented by independent displacement components.[10] In the same reference, the importance of these rigid body motions is clearly illustrated in several examples. However, the inclusion of rigid body motions was done at the expense of rigorous interelement compatibility. This compromise resulted in a significant improvement in the behavior of the element. In this paper we develop a method to include rigid body motions without comprising deformational compatibility. The method is general and can be applied without difficulties to any element, curved or flat. The improvements of a curved cylindrical element are illustrated with one example.

Stiffness Matrix

We presume that a stiffness matrix is constructed using a method that will insure that the required degree of interelement continuity is going to be maintained and that completeness of the straining modes is also going to be satisfied. The method developed by Bogner, Fox, and Schmit[8] is such a method. For the element shown in Fig. 1 we get a (24 × 24) stiffness matrix $[K]$ when we use the following nodal coordinates:

$$\{u_i\}^T = \langle u_i, v_i, w_i, w_{i,\xi}, \theta_i, w_{i,\xi\eta} \rangle i = 1,2,3,4 \tag{1}$$

where $\theta_i = w_{i,\eta} - v_i/r$. Novoshilov's theory was used to construct the matrix (13). This stiffness matrix will be referred to as CS1 in the following discussion. For axisymmetric elements the method described in Ref. 6 can be used. Once a stiffness matrix is obtained the number of strain free modes can be found by an eigenvalue analysis. For CS1 we find two strain free modes only.

Rigid Body Modes

Expressions for the coupled form of the curved displacement functions under all six rigid body motions of small amplitude are easily obtained from simple kinematics. The results for CS1 are taken from Ref. 10

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & -r(\cos\phi - \cos\beta) & -r\sin\phi \\ 0 & \cos\phi & -\sin\phi & -r(1 - \cos\phi\cos\beta) & -\xi\sin\phi & \xi\cos\phi \\ 0 & \sin\phi & \cos\phi & r\cos\beta\sin\phi & \xi\cos\phi & \xi\sin\phi \end{bmatrix} \begin{Bmatrix} S_x \\ S_y \\ S_z \\ \theta_x \\ \theta_y \\ \theta_z \end{Bmatrix} \tag{2}$$

where S_x, Sy, S_z, θ_x, θ_y, θ_z, are, respectively, the three components of a general rigid body translation and the three components of a general rigid body rotation of small amplitude in the system of reference (x,y,z), (see Fig. 1). For other stiffness matrices the same analysis applies, expressions like Eq. (2) would result.

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Table 1 Transformation matrix $[T]$[a]

	S_x	S_y	S_z	θ_x	θ_y	θ_z
u_1	1	0	0	0	0	$r \sin\beta$
v_1	0	$\cos\beta$	$\sin\beta$	$-r \sin^2\beta$	$-(a/2) \sin\beta$	$-(a/2) \cos\beta$
w_1	0	$-\sin\beta$	$\cos\beta$	$-r \sin\beta \cos\beta$	$-(a/2) \cos\beta$	$(a/2) \sin\beta$
$w_{1,\xi}$	0	0	0	0	$\cos\beta$	$-\sin\beta$
θ_1	0	0	0	1	0	0
$w_{1,\xi\eta}$	0	0	0	0	$(1/r) \sin\beta$	$(1/r) \cos\beta$
u_2	1		0	0	0	$r \sin\beta$
v_2	0	$\cos\beta$	$\sin\beta$	$-r \sin^2\beta$	$(a/2) \sin\beta$	$(a/2) \cos\beta$
w_2	0	$-\sin\beta$	$\cos\beta$	$-r \sin\beta \cos\beta$	$(a/2) \cos\beta$	$-(a/2) \sin\beta$
$w_{2,\xi}$	0	0	0	0	$\cos\beta$	$-\sin\beta$
θ_2	0	0	0	1	0	0
$w_{2,\xi\eta}$	0	0	0	0	$(1/r) \sin\beta$	$(1/r) \cos\beta$
u_3	1	0	0	0	0	$-r \sin\beta$
v_3	0	$\cos\beta$	$-\sin\beta$	$-r \sin^2\beta$	$-(a/2) \sin\beta$	$(a/2) \cos\beta$
w_3	0	$\sin\beta$	$\cos\beta$	$r \sin\beta \cos\beta$	$(a/2) \cos\beta$	$(a/2) \sin\beta$
$w_{3,\xi}$	0	0	0	0	$\cos\beta$	$\sin\beta$
θ_3	0	0	0	1	0	0
$w_{3,\xi\eta}$	0	0	0	0	$-(1/r) \sin\beta$	$(1/r) \cos\beta$
u_4	1	0	0	0	0	$-r \sin\beta$
v_4	0	$\cos\beta$	$-\sin\beta$	$-r \sin^2\beta$	$(a/2) \sin\beta$	$-(a/2) \cos\beta$
w_4	0	$\sin\beta$	$\cos\beta$	$r \sin\beta \cos\beta$	$-(a/2) \cos\beta$	$-(a/2) \sin\beta$
$w_{4,\xi}$	0	0	0	0	$\cos\beta$	$\sin\beta$
θ_4	0	0	0	1	0	0
$w_{4,\xi\eta}$	0	0	0	0	$-(1/r) \sin\beta$	$(1/r) \cos\beta$

[a] All the symbols in this table are defined in Fig. 1.

Modified Stiffness Matrix

The number of degrees of freedom of the original matrix is now expanded to contain the six components of rigid body motions as shown below for CS1,

$$\{u_i\} = [U \,\vdots\, T] \begin{Bmatrix} \bar{u}_i \\ \hline u_j^R \end{Bmatrix} \tag{3}$$

where $\bar{u}_i$ is a modified set of nodal coordinates defined by an equation identical to Eq. (1),

$$\{u_j^R\} = \langle S_x, S_y, S_z, \theta_x, \theta_y, \theta_z \rangle^T$$

$[U]$ is a (24×24) unit matrix and $[T]$ is a (24×6) transformation matrix relating the nodal coordinates to the six components of rigid body motions. Matrix $[T]$ is given in Table 1. The stiffness matrix is then expanded by a congruent transformation to add six rigid body modes. In the case of CS1 we get

$$\begin{bmatrix} U \\ \hline T^T \end{bmatrix} [K] \quad [U \,\vdots\, T] = \begin{bmatrix} k_{11} & \vdots & k_{12} \\ \hline k_{21} & \vdots & k_{22} \end{bmatrix} \tag{4}$$

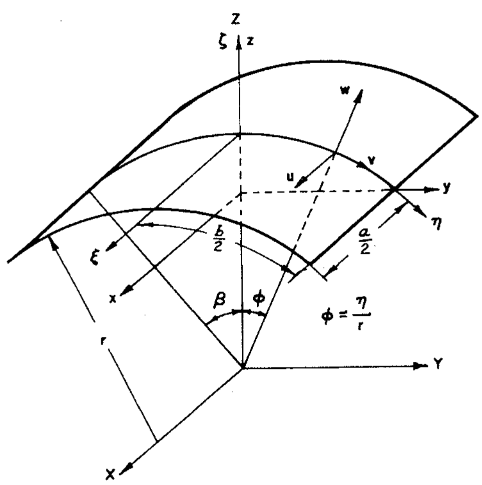

Fig. 1 Definition of geometrical symbols.

where

$$[k_{11}] = [K]; \; [k_{12}] = [k_{21}]^T = [K][T]$$

and

$$[k_{22}] = [T]^T[K][T]$$

Matrix $[k_{22}]$ is a (6×6) matrix, a typical example for CS1 is given in Table 2. Since rigid body modes should not introduce nodal forces, we set these forces equal to zero and write

$$\begin{Bmatrix} F \\ \hline 0 \end{Bmatrix} = \begin{bmatrix} k_{11} & \vdots & k_{12} \\ \hline k_{21} & \vdots & k_{22} \end{bmatrix} \begin{Bmatrix} \bar{u}_i \\ \hline u_j^R \end{Bmatrix} \tag{5}$$

We can now solve the last of Eqs. (5) for $\{u_j^R\}$,

$$\{u_j^R\} = -[k_{22}]^{-1}[k_{21}]\{\bar{u}_i\} \tag{6}$$

This last expression requires inverting $[k_{22}]$. This matrix, however, will be singular if the original stiffness matrix already contained any rigid body modes. A rapid examination of the numerical information contained in Table 2 shows that for CS1, S_x is already implicitly included in $[K]$; also we see that S_y and θ_x are linearly dependent. This is due to the fact that the method of construction of compatible displacements implicitly contains a rigid body curvilinear translation. We can now simply suppress S_x and S_y in $\{u_j^R\}$ and their corresponding columns in matrix $[T]$; matrix $[k_{22}]$ will now be a (4×4) diagonal matrix. Solving Eq. (6) and substituting in the

Fig. 2 Pinched cylinder: P = 100 lb; r = 4.953 in.; L = 10.35 in.; E = 10.5 × 10[6] psi; t = 0.094 in.; μ = 0.3125.

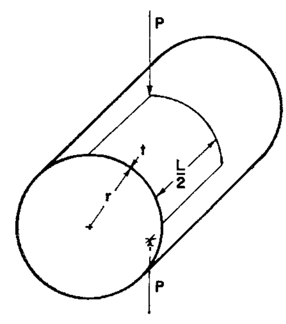

Table 2 Typical numerical example for matrix $[k_{22}]$[a]

	S_x	S_y	S_z	θ_x	θ_y	θ_z
S_x	0	0	0	0	0	0
S_y	0	0.0087083377	0	0.076422853	0	0
S_z	0	0	0.00014052739	0	0	0
θ_x	0	0.076422853	0	0.67067363	0	0
θ_y	0	0	0	0	0.29584907	0
θ_z	0	0	0	0	0	0.09797269

[a] The element data for this numerical example is as follows: $r = 10$, $a = 10$, $b = 10$, (see Fig. 1); Young's modulus = 1, Poisson's ratio = 0.3, thickness of the element = 0.1.

first of (5), we get

$$\{F\} = ([k_{11}] - [k_{12}][k_{22}]^{-1}[k_{21}])\{\bar{u}_i\} \tag{7}$$

The new modified stiffness matrix is then

$$[K^*] = [k_{11}] - [k_{12}][k_{22}]^{-1}[k_{21}] \tag{8}$$

This new matrix is now completely strain free for any conceivable rigid body motions, as it should be. A measure of the improvement of the properties of the new matrix can be taken as the reduction of its trace. In the example used in Table 2, the trace of the original matrix was 1.14738, the trace of the modified matrix is 0.9623086. The eigenvalues of both matrices are also given in Table 3. A more dramatic demonstration of the improvement of the matrix can be made by solving a problem particularly sensitive to rigid body motions. We again, as in Ref. 10, use a pinched cylinder for which results are available for an analysis using a (48 × 48) matrix in Ref. 11. The problem is illustrated in Fig. 2 and the results of our comparisons contained in Table 4. In this table, the results of a convergence study for the displacement under the load are also included. This problem has an approximate inextensional solution mentioned in Timoshenko[12] which gives a displacement of −0.1084 in. under the load; however this result is known to be too small. We believe that our result is correct to four significant digits as indicated in our convergence study. Other problems for which solutions were available by the Donnell-Jenkins theory gave the same results as those mentioned in Ref. 10.

Stresses and Strains

Using the original displacement functions, nodal strains can be obtained from relations similar to

$$\{\epsilon_i\} = [B]\{u_i\} \tag{9}$$

where $[B]$ is a numerical transformation matrix between the nodal strains and nodal coordinates. Using Eqs. (3) and (6) we obtain after substitution

$$\{\epsilon_i\} = [B]([U] - [T][k_{22}^{-1}][k_{21}])\{\bar{u}_i\} \tag{10}$$

or

$$\{\epsilon_i\} = [B^*]\{\bar{u}_i\} \tag{11}$$

where

$$[B^*] = [B]([U] - [T][k_{22}^{-1}][k_{21}]) \tag{12}$$

Compatibility

The original set of displacement functions used to construct a stiffness matrix can be written as

$$\{\tilde{u}\} = [P]\{u_i\} \tag{13}$$

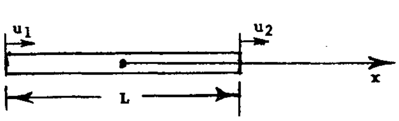

Fig. 3 One-dimensional element.

where $\{\tilde{u}\}$ is a vector of displacement components which in the case of the cylindrical element used here as an example is

$$\{\tilde{u}\} = \langle u,v,w\rangle^T \tag{14}$$

$[P]$ is an appropriate matrix of shape functions selected to satisfy interelement compatibility and $\{u_i\}$ is the unmodified set of nodal coordinates. In the present formulation, the set of nodal coordinates $\{u_i\}$ is modified at the element level by postulating that for any element, combinations of nodal coordinates proportional to the columns of matrix $[T]$ are going to give zero forces at the nodes. The essential character of (13) is not changed by replacing $\{u_i\}$ by its value from Eq. (3)

$$\{\tilde{u}\} = [P]\{\bar{u}_i\} + [P][T]\{u_j^R\} \tag{15}$$

In this form it is clear that the first term is going to give a continuous field because the modified coordinates are going to be continuous at the nodes. In the second term, however, the $\{u_j^R\}$ are going to vary from element to element, since rigid body motions may well be different in two adjacent elements. This second term represents an imperfect mapping of the true rigid body modes and does not preserve full compatibility along the edges of an element. However, since the $\{u_j^R\}$ are linearly dependent upon the $\{\bar{u}_i\}$, nodal continuity is preserved even for this term.

In the construction of consistent load vectors corresponding to body force fields, the author has used only the first term of Eq. (15). The results obtained in the analysis of several problems have been consistently better than the results obtained without modifications of the stiffness matrix.

Conclusions

The method described here to introduce all of the rigid body modes in a stiffness matrix that otherwise satisfies all other conditions to insure convergence to an acceptable result is both very simple of application and general. Curved elements corrected by this method show a significant improvement in the solution of problems requiring rigid motions of

Table 3 Eigenvalues of a typical numerical example for $[K]$ and $[K^*]$[a]

	K	K^*		K	K^*
1	5.709 D-01	5.455 D-01	13	3.371 D-03	1.401 D-03
2	1.325 D-01	1.303 D-01	14	2.395 D-03	4.593 D-04
3	1.107 D-01	9.899 D-02	15	1.477 D-03	3.755 D-04
4	1.003 D-01	6.090 D-02	16	1.401 D-03	1.947 D-04
5	7.697 D-02	4.615 D-02	17	1.031 D-03	5.360 D-05
6	4.504 D-02	2.597 D-02	18	2.227 D-04	4.896 D-06
7	3.204 D-02	2.396 D-02	19	2.129 D-04	2.876 D-17
8	2.328 D-02	1.076 D-02	20	1.071 D-04	6.891 D-18
9	1.833 D-02	6.049 D-03	21	6.290 D-07	4.877 D-18
10	1.533 D-02	4.770 D-03	22	1.817 D-07	4.059 D-20
11	6.431 D-03	4.359 D-03	23	3.027 D-17	−1.857 D-17
12	5.348 D-03	2.161 D-03	24	8.733 D-18	−2.181 D-17

[a] The data for the element used in this example is the same as in Table 2.

Table 4 Displacement under the load for the pinched cylinder

Bogner et al. (48 × 48)			Cantin [K*] (24 × 24)		
No. of Eq.	Mesh	Displac., in.	No. of Eq.	Mesh	Displac., in.
72	1 × 2	−0.0802	36	1 × 2	−0.0921
96	1 × 3	−0.1026	48	1 × 3	−0.1072
120	1 × 4	−0.1087	60	1 × 4	−0.1099
108	2 × 2	−0.0808	54	2 × 2	−0.0931
144	2 × 3	−0.1036	72	2 × 3	−0.1085
180	2 × 4	−0.1098	90	2 × 4	−0.1113
			54	2 × 2	−0.0931
			150	4 × 4	−0.1126
			294	6 × 6	−0.1137
			486	8 × 8	−0.1139
			726	10 × 10	−0.1139

some elements. The problems analysed with this stiffness matrix converged while the mesh was still coarse enough to allow substantial curvature in all elements.

Appendix

The following problem has been worked out by professor R. E. Newton of the Naval Postgraduate School and is included here as a complete example of the modifications proposed to improve a stiffness matrix.

Consider a straight one-dimensional element of constant cross section (A), with two nodal points. Assume the following displacement functions: (see Fig.3)

$$u(x) = \frac{(L - 2x)^2}{4L^2} u_1 + \frac{(L + 2x)^2}{4L^2} u_2$$

where L is the length of the element. Using the standard procedures with $\epsilon(x) = du/dx$ and $\sigma = E\epsilon$, we obtain the following results for the unmodified stiffness and $[B]$ matrices:

$$[K] = \frac{2AE}{3L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \text{ and } [B] = \frac{2}{L}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Transformation matrix $[T]$ in this case is a single column: $\{T\} = \langle 1,1 \rangle^T$. Matrices $[K]$ and $[B]$ are patently erroneous, however, after the corrections proposed here the results are

$$[K^*] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ and } [B^*] = \frac{1}{L}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

and these results are exact, for concentrated nodal forces.

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